

A 2-COCYCLE ON A GROUP OF SYMPLECTOMORPHISMS

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ABSTRACT. For a symplectic manifold (M, ω) with exact symplectic form we construct a 2-cocycle on the group of symplectomorphisms and indicate cases when this cocycle is not trivial.

1. INTRODUCTION

For a symplectic manifold (M, ω) such that $H^1(M, \mathbb{R}) = 0$ and the symplectic form ω is exact we indicate a formula defining a 2-cocycle on the group $\text{Diff}(M, \omega)$ of symplectomorphisms with values in the trivial $\text{Diff}(M, \omega)$ -module \mathbb{R} . Let G be a connected real simple Lie group and K a maximal compact subgroup. For the symmetric Hermitian space $M = G/K$ endowed with the induced symplectic structure, we prove that the restriction of this cocycle to the group G is non-trivial. Thus this cocycle is non-trivial on the whole group $\text{Diff}(M, \omega)$, too. In particular, this implies that the cocycle is non-trivial for the symplectic manifold $(\mathbb{R}^2 \times M, \omega_0 + \omega_M)$, where (M, ω_M) is a non-compact symplectic manifold with exact symplectic form ω_M such that $H^1(M, \mathbb{R}) = 0$ and ω_0 is the standard symplectic form on \mathbb{R}^2 .

For the convenience of the reader, in an appendix we consider the corresponding 2-cocycle on the Lie algebra of locally Hamiltonian and Hamiltonian vector fields and indicate when this cocycle is non-trivial.

Note that in [7] a similar 2-cocycle was constructed for the group of volume preserving diffeomorphisms on a compact n -dimensional manifold M . This cocycle takes its values in the space $H^{n-2}(M, \mathbb{R})$. Neretin in [10] constructed a 2-cocycle on the group of symplectomorphisms with compact supports.

Throughout the paper M is a connected C^∞ -manifold.

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2. PRELIMINARIES

We recall some standard facts on central extensions of groups and two-dimensional cohomology of groups (see, for example, [8], ch. 4).

Consider a group G and the field \mathbb{R} as a trivial G -module. Let $C^p(G, \mathbb{R})$ be the set of maps from G^p to \mathbb{R} for $p > 0$ and let $C^0(G, \mathbb{R}) = \mathbb{R}$. Define a map $D^p : C^p(G, \mathbb{R}) \rightarrow C^{p+1}(G, \mathbb{R})$ as follows: for $f \in C^p(G, \mathbb{R})$ and $g_1, \dots, g_{p+1} \in G$

$$(1) \quad (D^p f)(g_1, \dots, g_{p+1}) = f(g_2, \dots, g_{p+1}) + \sum_{i=1}^p (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{p+1}) + (-1)^{p+1} f(g_1, \dots, g_p).$$

By definition $C^*(G, \mathbb{R}) = (C^p(G, \mathbb{R}), D^p)_{p \geq 0}$ is the standard complex of nonhomogeneous cochains of the group G with values in the G -module \mathbb{R} and its cohomology $H^*(G, \mathbb{R}) = (H^p(G, \mathbb{R}))_{p \geq 0}$ is the cohomology of the group G with values in the trivial G -module \mathbb{R} . Recall that a cochain $f \in C(G, \mathbb{R})$ is called normalized if $f(g_1, \dots, g_p) = 0$ whenever at least one of the $g_1, \dots, g_p \in G$ equals the identity e of G . It is known that the inclusion of the subcomplex of normalized cochains into $C^*(G, \mathbb{R})$ induces an isomorphism in cohomology.

Let f be a normalized 2-cocycle of G with values in a trivial G -module \mathbb{R} . Let $E(G, \mathbb{R}) = G \times \mathbb{R}$, with multiplication $(g_1, a_1)(g_2, a_2) = (g_1 g_2, a_1 g_2 + a_2 + f(g_1, g_2))$ for $a_1, a_2 \in \mathbb{R}$ and $g_1, g_2 \in G$. Then $E(G, \mathbb{R})$ is a group, and the natural projection $E(G, \mathbb{R}) = G \times \mathbb{R} \rightarrow G$ is a central extension of the group G by \mathbb{R} . The extension $E(G, \mathbb{R})$ is non-split iff the cocycle f is non-trivial.

If G is a topological group (finite-dimensional or infinite-dimensional Lie group) one can define a subcomplex $C_{\text{cont}}^*(G, \mathbb{R})$ ($C_{\text{diff}}^*(G, \mathbb{R})$) of the complex $C^*(G, \mathbb{R})$ (see [4], ch. 3) consisting of cochains which are continuous (smooth) functions. The cohomologies of the complexes $C_{\text{cont}}^*(G, \mathbb{R})$ and $C_{\text{diff}}^*(G, \mathbb{R})$ are isomorphic whenever G is a finite-dimensional Lie group (see [4], ch. 3 and [9]). Note that if the 2-cocycle f is continuous (differentiable), the extension $E(G, \mathbb{R})$ is a topological group (Lie group).

3. A 2-COCYCLE ON THE GROUP OF SYMPLECTOMORPHISMS

Let (M, ω) be a non-compact symplectic manifold such that $H^1(M, \mathbb{R}) = 0$ and the symplectic form ω is exact. Let ω_1 be a 1-form on M such that $d\omega_1 = \omega$. Denote by $\text{Diff}(M, \omega)$ the group of symplectomorphisms of M . We define a 2-cocycle on the group $G = \text{Diff}(M, \omega)$ with values in the trivial G -module \mathbb{R} as follows. Fix a point $x_0 \in M$. Then for $g_1, g_2 \in G$ we put

$$(2) \quad C_{x_0}(g_1, g_2) = \int_{x_0}^{g_2 x_0} (g_1^* \omega_1 - \omega_1),$$

where the integral is taken along a smooth curve connecting the point x_0 with the point g_2x_0 . Since $H^1(M, \mathbb{R}) = 0$ the 1-form $g_1^*\omega_1 - \omega_1$ is exact and the value of this integral does not depend on the choice of such a curve.

Theorem 3.1. *The function $C_{x_0} : G^2 \rightarrow \mathbb{R}$ defined by (2) is a normalized 2-cocycle on the group G with values in the trivial G -module \mathbb{R} . The cohomology class of C_{x_0} is independent of the choice of the point x_0 and the form ω_1 .*

Proof. By (1) it is easy to check that $D^2C_{x_0} = 0$. Moreover, the 2-cocycle C_{x_0} is normalized. Since for each $g \in G$ the 1-form $g^*\omega_1 - \omega_1$ is exact, for any points $x_1, x_2 \in M$ we have $C_{x_1} - C_{x_2} = Da$, where a is a 1-cochain on G defined by $a(g) = \int_{x_1}^{x_2} (g^*\omega_1 - \omega_1)$. \square

By definition, the cocycle C_{x_0} is a continuous function on $G \times G$.

Remark 3.2. Let M be a manifold such that $H^1(M, \mathbb{R}) = 0$ and let ω be an exact 2-form on M . Let $\text{Diff}(M, \omega)$ be the group of diffeomorphisms of M preserving the form ω . Then the formula (2) for $g_1, g_2 \in \text{Diff}(M, \omega)$ gives a 2-cocycle on the group $\text{Diff}(M, \omega)$ and all statements of theorem (3.1) are true for this cocycle.

Denote by $E(\text{Diff}(M, \omega))$ the central extension of the group $\text{Diff}(M, \omega)$ by \mathbb{R} defined by the cocycle C_{x_0} . Now we give a geometric interpretation of the extension $E(\text{Diff}(M, \omega))$. We choose a form ω_1 with $d\omega_1 = \omega$ and put $\omega_2(g) = \int_{x_0}^x (\omega_1 - g^*\omega_1)$. Consider the trivial \mathbb{R} -bundle $M \times \mathbb{R}$. Clearly, the form $dt + \omega_1$ is a connection with curvature ω of this bundle. Denote by $\text{Aut}(M \times \mathbb{R}, \omega)$ the group of those bundle automorphisms which respect the connection $dt + \omega_1$ and which are projectable to diffeomorphisms in $\text{Diff}(M, \omega)$. It is easy to check that the group $\text{Aut}(M \times \mathbb{R}, \omega)$ is isomorphic to the group $E(\text{Diff}(M, \omega)) = \text{Diff}(M, \omega) \times \mathbb{R}$ which acts as follows on $M \times \mathbb{R}$: $(x, t) \rightarrow (g(x), \omega_2(g)(x) + t + a)$, where $(x, t) \in M \times \mathbb{R}$ and $(g, a) \in G \times \mathbb{R}$. This gives an equivalent definition of the extension $E(\text{Diff}(M, \omega))$ as a group of automorphisms of the trivial principal \mathbb{R} -bundle $M \times \mathbb{R}$ with connection $dt + \omega_1$.

If we replace the form ω_1 by the form $\omega_1 + df$, where f is a smooth function on M , we get an action of G on $M \times \mathbb{R}$ which is related to the initial one by the gauge transformation $(x, t) \rightarrow (x, t - f(x))$ of the bundle $M \times \mathbb{R} \rightarrow M$.

4. EXAMPLES OF NON-TRIVIAL 2-COCYCLES

The authors are not able to prove that the cocycle C_{x_0} is non-trivial for any symplectic manifold M with an exact symplectic 2-form ω . In this section we prove that for some symplectic manifolds the restrictions of this cocycle to some subgroups of $G \subset \text{Diff}(M, \omega)$ turn out to be non-trivial.

4.1. The linear symplectic space \mathbb{R}^{2n} and the Heisenberg group. Consider the space \mathbb{R}^{2n} with the standard symplectic form $\omega_0 = \sum_{k=1}^n dx_k \wedge dx_{k+n}$ and the group $G = \mathbb{R}^{2n}$ acting on the space \mathbb{R}^{2n} by translations. Applying (2) to the form

ω_0 , the 1-form $\omega_1 = \frac{1}{2} \sum_{k=1}^n (x_{n+k} dx_k - x_k dx_{n+k})$, and the point $x_0 = 0 \in \mathbb{R}^{2n}$ we get a 2-cocycle on the group G given by

$$C_0(x, y) = \frac{1}{2} \sum_{k=1}^n (x_k y_{n+k} - y_k x_{n+k}),$$

where $x = (x_1, \dots, x_{2n})$ and $y = (y_1, \dots, y_{2n})$. The central extension of the group \mathbb{R}^{2n} by \mathbb{R} defined by this cocycle is the Heisenberg group. This extension is non-split since the Heisenberg group is noncommutative and thus the cocycle $C_0(x, y)$ is non-trivial.

4.2. Symmetric Hermitian spaces and the Guichardet-Wigner cocycle. Consider a non-compact symmetric space $M = G/K$, where G is a connected real simple Lie group and where K is a maximal compact subgroup. Then M is diffeomorphic to \mathbb{R}^n , where $n = \dim M$. We suppose that M admits a G -invariant complex structure, i.e., M is a symmetric Hermitian space. This condition is satisfied (up to finite covering) for the following groups: $SU(p, q)$ ($p, q \geq 1$), $SO_0(2, q)$ ($q = 1$ or $q \geq 3$), $Sp(n, \mathbb{R})$ ($n \geq 1$), $SO^*(2n)$ ($n \geq 2$), and certain real forms of E_6 and E_7 .

Consider the symplectic manifold (M, ω) , where the symplectic form ω is defined by the Hermitian metric on M . It is known that on each of the Lie groups mentioned above, in the complex $C_{\text{diff}}(G, \mathbb{R})$ there is a non-trivial Guichardet-Wigner 2-cocycle (see [5] and [4]). By [2] this cocycle is given as follows, up to a nonzero factor:

$$(3) \quad (g_1, g_2) \mapsto \int_{(x_0, g_1 x_0, g_1 g_2 x_0)} \omega,$$

where $g_1, g_2 \in G$, $x_0 \in K \subset G/K$, and the integral is taken over the oriented geodesic cone with vertex x_0 and the segment of a geodesic from $g_1 x_0$ to $g_1 g_2 x_0$ as base.

We prove that the restriction of the cocycle C_{x_0} to the group G is cohomologous to the cocycle given by (3).

For the base point x_0 we define a 1-cochain γ_{x_0} on the group G as follows:

$$\gamma_{x_0}(g) = \int_{x_0}^{gx_0} \omega_1,$$

where $g \in G$ and the integral is taken along the geodesic segment from x_0 to gx_0 . Consider C_{x_0} on G given by formula (2), where we choose for the curve between the points x_0 and $g_2 x_0$ a geodesic segment from x_0 to $g_2 x_0$. It is easy to check that on the group G the cocycle $C_{x_0} + D\gamma_{x_0}$ equals the cocycle given by (3). Thus the cocycle C_{x_0} on the group G is non-trivial in the complex $C_{\text{diff}}^*(G, \mathbb{R})$.

In particular, for the group $G = \text{SL}(2, \mathbb{R}) = \text{SU}(1, 1)$ the symmetric space $M = G/K$ is the hyperbolic plane H^2 and ω is the area form on H^2 . Instead of the group $\text{SL}(2, \mathbb{R})$ we will later consider the group $\text{PSL}(2, \mathbb{R})$ which acts effectively on H^2 . Since $\text{SL}(2, \mathbb{R})$ is a two-sheet cover of $\text{PSL}(2, \mathbb{R})$, the cohomologies of these groups

with values in \mathbb{R} are the same. It is easy to check that the corresponding symplectic manifold (M, ω) is isomorphic to the symplectic manifold (\mathbb{R}^2, ω_0) , where ω_0 is the standard symplectic form on \mathbb{R}^2 . Unfortunately, for the groups $G \neq \mathrm{SU}(1, 1)$ mentioned above we do not know whether the symplectic manifolds (M, ω) and $(\mathbb{R}^{2n}, \omega_0)$, where $\dim M = 2n$, are isomorphic or not.

The following proposition is known. We do not know a good reference for this; then we give a short proof communicated to us by Yu.A. Neretin.

Proposition 4.3. *For each symmetric Hermitian space $M = G/K$, where G is a connected simple Lie group and K is its maximal compact subgroup, the corresponding Guichardet-Wigner cocycle is non-trivial in the complex $C^*(G, \mathbb{R})$.*

Proof. Let $p : \tilde{G} \rightarrow G$ be the universal cover and let $a = C_{x_0}$ be the Guichardet-Wigner cocycle for the group G . Consider the corresponding to a 2-cocycle \tilde{a} on \tilde{G} induced by p . By construction, the cocycle \tilde{a} is trivial, i.e., there is a smooth function b defined on \tilde{G} such that for any $g, h \in \tilde{G}$ we have $\tilde{a}(g, h) = b(h) - b(gh) + b(g)$.

Assume that the cocycle a is trivial in the complex $C^*(G, \mathbb{R})$, i.e., there exists a function $f : G \rightarrow \mathbb{R}$ such that for $g, h \in \Gamma$ we have $a(g, h) = f(h) - f(gh) + f(g)$.

Then the difference $b - f \circ p$ is a homomorphism $\tilde{G} \rightarrow \mathbb{R}$. This homomorphism vanishes near the identity element of \tilde{G} since the group \tilde{G} is simple, and thus it vanishes on the whole of \tilde{G} since \tilde{G} is connected. Then the function f is smooth and the cocycle a is trivial in the complex $C_{\mathrm{diff}}^*(G, \mathbb{R})$. This contradiction proves our statement. \square

5. CASES OF NONTRIVIALITY OF THE COCYCLE C_{x_0} FOR GROUPS OF SYMPLECTOMORPHISMS

Let (M, ω_M) be a non-compact symplectic manifold such that $H^1(M, \mathbb{R}) = 0$ with an exact symplectic form ω_M .

By formula (2), the form ω_M defines a 2-cocycle C_{x_0} for the group $\mathrm{Diff}(M, \omega_M)$ with values in the trivial $\mathrm{Diff}(M, \omega_M)$ -module \mathbb{R} . The aim of this section is to indicate cases when this cocycle is non-trivial and thus the corresponding central extension of the group $\mathrm{Diff}(M, \omega_M)$ by \mathbb{R} is non-split.

Let $M = G/K$ be an Hermitian symmetric space M and let (M, ω) be the corresponding symplectic manifold which we considered in subsection 4.2.

Theorem 5.1. *For the Hermitian symmetric space $M = G/K$ and for the corresponding symplectic manifold (M, ω) the cocycle C_0 on the group $\mathrm{Diff}(M, \omega)$ is non-trivial.*

Proof. Since the group G is a subgroup of the group $\mathrm{Diff}(M, \omega)$ the statement follows from proposition 4.3. \square

Recall that the symplectic manifold (H^2, ω) where ω is the area form is symplectomorphic to (\mathbb{R}^2, ω_0) where ω_0 is the standard symplectic form.

Theorem 5.2. *Let (M, ω) be a non-compact symplectic manifold such that the symplectic form ω_M is exact and let $H^1(M, \mathbb{R}) = 0$. Consider the product $\mathbb{R}^2 \times M$ of the manifold \mathbb{R}^2 and M as a symplectic manifold with the symplectic form $\omega = \omega_0 + \omega_M$. Then for each point $x_0 \in \mathbb{R}^2 \times M$ the cocycle C_{x_0} on the group $\text{Diff}(\mathbb{R}^2 \times M, \omega)$ is non-trivial.*

Proof. Choose $\omega_{M,1} \in \Omega^1(M)$ with $d\omega_{M,1} = \omega_M$ and let $\omega_1 = x dy + \omega_{M,1}$. The group $\text{Diff}(\mathbb{R}^2, \omega_0)$ acting on the first factor \mathbb{R}^2 of $\mathbb{R}^2 \times M$ is naturally included as a subgroup into the group $\text{Diff}(\mathbb{R}^2 \times M, \omega)$. Thus $g^*\omega_1 - \omega_1 = g^*(x dy) - x dy$ for all g in the subgroup $\text{Diff}(\mathbb{R}^2, \omega_0)$. Thus the cocycle C_{x_0} constructed from the form $dx \wedge dy + \omega_M$ on $\mathbb{R}^2 \times M$ restricts to a nontrivial cocycle on the subgroup of $\text{Diff}(\mathbb{R}^2, \omega_0)$ by proposition 4.3 applied to the group $\text{PSL}(2, \mathbb{R})$. \square

We leave to the reader to formulate the corresponding results for other symmetric Hermitian spaces G/K instead of H^2 .

5.3. Problem. Consider an open disk M in the Euclidean plane equipped with the standard area 2-form ω . Is the 2-cocycle C_{x_0} defined by the form ω non-trivial?

6. APPENDIX

In this appendix, for a symplectic manifold (M, ω) we define a 2-cocycle on the Lie algebra $\text{Vect}(M, \omega)$ of locally Hamiltonian or Hamiltonian vector fields, corresponding to the 2-cocycle C_{x_0} on the group $\text{Diff}(M, \omega)$, and study conditions of its nontriviality.

Let \mathfrak{g} be a Lie algebra over \mathbb{R} and let \mathbb{R} be the trivial \mathfrak{g} -module. Denote by $C^p(\mathfrak{g}, \mathbb{R})$ the space of skew-symmetric p -forms on \mathfrak{g} with values in \mathbb{R} . For $c \in C^p(\mathfrak{g}, \mathbb{R})$ and x_1, \dots, x_{p+1} put

$$(4) \quad (\delta^p c)(x_1, \dots, x_{p+1}) = \sum_{i < j} (-1)^{i+j} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1}),$$

where, as usual, \hat{x} means that x is omitted. Then $C^*(\mathfrak{g}, \mathbb{R}) = (C^p(\mathfrak{g}, \mathbb{R}), \delta^p)_{p \geq 0}$ is the complex of standard cochains of the Lie algebra \mathfrak{g} with values in the trivial \mathfrak{g} -module \mathbb{R} and the cohomology $H^*(\mathfrak{g}, \mathbb{R})$ of this complex is the cohomology of the Lie algebra \mathfrak{g} with values in the trivial \mathfrak{g} -module \mathbb{R} .

In particular, there is a bijective correspondence between $H^2(\mathfrak{g}, \mathbb{R})$ and the set of isomorphism classes of central extensions of the Lie algebra \mathfrak{g} by \mathbb{R} .

Let (M, ω) be a symplectic manifold. Denote by $\text{Vect}(M, \omega)$ the Lie algebra of locally Hamiltonian vector fields and by $\text{Vect}_0(M, \omega)$ the Lie algebra of Hamiltonian vector fields on M . For a point $x_0 \in M$ and $X, Y \in \text{Vect}(M, \omega)$ put $c_{x_0}(X, Y) = \omega(X, Y)(x_0)$.

Proposition 6.1. *The function $c_{x_0} : \mathfrak{g}^2 \rightarrow \mathbb{R}$ is a 2-cocycle on the Lie algebra \mathfrak{g} with values in the trivial \mathfrak{g} -module \mathbb{R} . The cohomology class of c_{x_0} is independent of the choice of the point x_0 .*

Proof. The proof is given by direct calculations and is based on the standard formulas $[\mathcal{L}_X, \mathbf{i}_Y] = \mathbf{i}_{[X,Y]}$ and $\mathcal{L}_X = \mathbf{i}_X d + d\mathbf{i}_X$, where \mathbf{i}_X is the operator of the inner product by X and \mathcal{L}_X is the Lie derivative with respect to a vector field X , (see, for example, [6], ch. 4). In particular, we have for any $x \in M$ and $X, Y \in \text{Vect}(M, \omega)$ the following equality

$$(5) \quad c_x(X, Y) - c_{x_0}(X, Y) = - \int_{x_0}^x \mathbf{i}_{[X,Y]} \omega.$$

□

Let G be a Lie group and let \mathfrak{g} be its Lie algebra. We have a natural homomorphism of complexes $C_{\text{diff}}^*(G, \mathbb{R}) \rightarrow C^*(\mathfrak{g}, \mathbb{R})$ (see, for example, [4], ch. 3). In particular, if $c \in C_{\text{diff}}^2(G, \mathbb{R})$, the corresponding cochain $\tilde{c} \in C^2(\mathfrak{g}, \mathbb{R})$ is defined as follows:

$$\tilde{c}(X, Y) = \frac{\partial^2}{\partial t \partial s} (c(\exp tX, \exp sY) - c(\exp sY, \exp tX))_{t=0, s=0}$$

where $X, Y \in \mathfrak{g}$.

Let G be a Lie group of diffeomorphisms of M contained in the group $\text{Diff}(M, \omega)$. Then for the 2-cocycle $c = C_{x_0}$ of section 3, the cocycle \tilde{c} is cohomologous to the restriction of the cocycle c_{x_0} to the Lie algebra \mathfrak{g} of G . Unfortunately, we cannot apply this procedure to the whole group $\text{Diff}(M, \omega)$ and the Lie algebra $\text{Vect}(M, \omega)$. Therefore, the problems of nontriviality of 2-cocycles C_{x_0} on the group $\text{Diff}(M, \omega)$ and c_{x_0} on the Lie algebra $\text{Vect}(M, \omega)$ should be solved independently.

For each $X \in \text{Vect}(M, \omega)$ denote by α_X the closed 1-form such that $\alpha_X = \mathbf{i}_X \omega$. For all vector fields $X, Y \in \text{Vect}(M, \omega)$ we have the following equality:

$$(6) \quad \omega(X, Y) \omega^n = n \alpha_X \wedge \alpha_Y \wedge \omega^{n-1}$$

which can be easily checked in Darboux coordinates.

Denote by X_f a Hamiltonian vector field defined by a function $f \in C^\infty(M)$. Consider the Poisson algebra $P(M) = P(M, \omega)$ on (M, ω) , i.e., the algebra $C^\infty(M)$ endowed with the Poisson bracket $\{f, g\} = -\omega(X_f, X_g)$ for $f, g \in C^\infty(M)$.

The map $P(M) \rightarrow \text{Vect}_0(M, \omega)$ given by $f \rightarrow X_f$ is a homomorphism of Lie algebras which defines an extension of $\text{Vect}_0(M, \omega)$ by \mathbb{R} . It is easy to check that this extension is isomorphic to one given by the cocycle $-c_{x_0}$.

Theorem 6.2. *For a non-compact symplectic manifold (M, ω) the cocycle c_{x_0} on the Lie algebras $\text{Vect}(M, \omega)$ and $\text{Vect}_0(M, \omega)$ is non-trivial.*

Proof. It suffices to prove our statement for the Lie algebra $\text{Vect}_0(M, \omega)$.

First we prove that for each form $\beta \in \Omega^{2n-1}(M)$ there is a unique form $\alpha \in \Omega^1(M)$ such that $\beta = \alpha \wedge \omega^{n-1}$. Indeed, using Darboux coordinates it is easy to check that this has a unique local solution α . These are compatible and we get a global solution by gluing them.

Note that for each form $\alpha \in \Omega^1(M)$ there is a positive integer N and $2N$ functions $f_k, g_k \in C^\infty(M)$ ($k = 1, \dots, N$) such that $\alpha = \sum_{k=1}^N f_k dg_k$ which follows easily from the existence (by dimension theory) of a finite atlas for M .

Since $H^{2n}(M, \mathbb{R}) = 0$ there is a form $\beta \in \Omega^{2n-1}(M)$ such that $\omega^n = d\beta$. Then we have $\omega^n = \sum_{k=1}^N df_k \wedge dg_k \wedge \omega^{n-1}$. By (6) and using this equality we get

$$(7) \quad \sum_{k=1}^N \{f_k, g_k\} = -n.$$

Assume that the extension $P(M) \rightarrow \text{Vect}_0(M, \omega)$ $P(M) \rightarrow \text{Vect}_0(M, \omega)$ is split. Then $P(M)$ is a direct sum of the space of constant functions on M and an ideal isomorphic to $\text{Vect}_0(M, \omega)$ by $P(M) \rightarrow \text{Vect}_0(M, \omega)$. Equality (7) means that these summands have nonzero intersection. This contradiction proves the statement. \square

Now we consider a compact symplectic manifold (M, ω) . It is known that the extension $P(M) \rightarrow \text{Vect}_0(M, \omega)$ is split.

For a closed form α denote by $[\alpha]$ the cohomology class of α . Denote by L the linear map $H^p(M, \mathbb{R}) \rightarrow H^{p+2}(M, \mathbb{R})$ defined by $a \rightarrow a \cup [\omega]$, where $a \in H^p(M, \mathbb{R})$.

Theorem 6.3. *Let (M, ω) be a compact symplectic manifold. The cocycle c_{x_0} on the Lie algebra $\text{Vect}(M, \omega)$ is non-trivial iff the linear map*

$$L^{n-1} : H^1(M, \mathbb{R}) \rightarrow H^{2n-1}(M, \mathbb{R})$$

is not equal zero.

Proof. We may assume that $\int_M \omega^n = 1$. Put for brevity $V = \text{Vect}(M, \omega)$ and $V_0 = \text{Vect}_0(M, \omega)$. Set

$$b(X, Y) = \int_M \omega(X, Y) \omega^n,$$

where $X, Y \in V$. It is easy to check that b is a 2-cocycle on V .

Multiplying both sides of equality (5) by ω^n and integrating over M we get

$$(8) \quad b(X, Y) - c_{x_0}(X, Y) = \int_M \left(\int_{x_0}^x \mathbf{i}_{[X, Y]} \omega \right) \omega^n.$$

Since the right hand side of (8) is a coboundary of a 1-cochain in $C^1(V, \mathbb{R})$, the cocycles c_{x_0} and b are cohomologous. By (6) we have

$$(9) \quad b(X, Y) = n \int_M \alpha_X \wedge \alpha_Y \wedge \omega^{n-1},$$

for any $X, Y \in V$. If $X \in V_0$ the form α_X is exact, and $b(X, Y) = 0$ by (9). This proves (1).

Suppose that the cocycle b is trivial, i.e., there is a linear functional f on V such that for any $X, Y \in V$ we have $b(X, Y) = f([X, Y])$. By [1] we have $[V, V] = [V_0, V_0] = V_0$. This implies $b = 0$. So the cocycle b is trivial iff it equals zero. By (9) and the Poincaré duality this implies that $L^{n-1} = 0$ on $H^1(M, \mathbb{R})$. This proves (2). \square

We know no example when $H^1(M, \mathbb{R}) \neq 0$ and the map $L^{n-1} = 0$. Moreover, if M is a compact Kählerian manifold the map $L^{n-1} : H^1(M, \mathbb{R}) \rightarrow H^{2n-1}(M, \mathbb{R})$ is an isomorphism (see, for example, [11], ch. 4). Thus in this case the cocycle c_{x_0} is non-trivial whenever $H^1(M, \mathbb{R}) \neq 0$.

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